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On Some Classes of Analytic Functions Defined by Subordination

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Abstract - In this paper , we define some general classes of analytic functions by subordination . Our new results extend and improve a lot of known results (see [6]).

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On Some Classes of Analytic Functions Defined by Subordination

A. El-Sayed Ahmed

Abstract - In this paper , we define some general classes of analytic functions by subordination . Our new results extend and improve a lot of known results (see [6]).

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I. Introduction

Let A be the class of functions f which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and are given by

$$f(z) = 1 + \sum_{n=k}^{\infty} a_k z^k, \qquad n \in N. \tag{1.1}$$

A function f analytic in Δ is said to be univalent in a domain D if

$$f(z_1) = f(z_2) \Longrightarrow z_1 = z_2 \qquad z_1, z_2 \in D.$$

The class of all univalent functions f in Δ and have form (1.1) will be denoted by S.

A domain D is called convex if for every pair of points w_1 and w_2 in the interior of D, the line-segment joining w_1 to w_2 lies wholly in D. A function f which maps Δ onto a convex domain is called a convex function.The necessary and sufficient condition for $f \in S$ to be convex in Δ is that $\mathrm{Re}\frac{(zf'(z))'}{f'(z)} > 0, \quad z \in \Delta$. The class off all functions convex and univalent in Δ is denoted by C.

A domain D is said to be starlike with respect to $w = 0$ if the linear segment joining $w = 0$ to any other point of D lies wholly in D. If a function f maps Δ onto a starlike domain with respect to $w = 0$, then f is said to be starlike. The necessary and sufficient condition for $f \in S$ to be starlike is that

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0, \quad z \in \Delta.$$

This class is denoted by S^*, and it was studied first by Alexander [3].

Let $f(z)$ and $g(z)$ be analytic in Δ. We say that $f(z)$ is subordinate to $g(z)$ if there exists a function $\phi(z)$ analytic (not necessarily univalent) in Δ satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

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Ref.

[6] Z.M.G. Kishka, On some Classes of analytic functions, Bulletin de la Societe' Royale des Sciences de Lie'ge, 62(5-6)(1993), 313-360.

[3] J. Alexander , Functions with map the interior of the unit circle upon simple regions, Ann. Math. 17(1915-1916),12-22.

$$f(z) = g(\phi(z)) \quad (|z| < 1). \tag{1.2}$$

Subordination is denoted by $f(z) \prec g(z)$. For more details on univalent functions by subordination, we refer to [1,2,5,7-16].

Let B be the class of functions, analytic in Δ and of the form

$$w(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad n \in N, \tag{1.3}$$

and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$. Based on the class B Janowski [4] defined the class $P[A, B]$, as follows:

Let p be analytic function in Δ, given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{1.4}$$

Then $p(z)$ is said to be in the class $P[A, B]$; $-1 \leq B < A \leq 1$; if and only if, for $z \in \Delta$

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \qquad ; w \in B. \tag{1.5}$$

Concerning the class $P[A, B]$ Janowski [4] proved the following lemma:

Lemma 1.1 [4]. Let $p \in P[A, B]$, and given by (1.4). Then

(i) $-|p_n| \leq A - B$,

(ii) $-\frac{1-Ar}{1-Br} \leq \text{Re p(z)} \leq \frac{1+Ar}{1+Br}$,

(iii) $-|\arg \text{p(z)}| \leq \sin^{-1} \frac{(A-B)r}{1-ABr^2}$

These results are sharp.

Let N and D be analytic in Δ, D maps Δ onto a many -sheeted starlike region, $N(0) = D(0)$, and

$$\frac{N'(z)}{D'(z)} \in P[A, B], \qquad \text{then} \quad \frac{N(z)}{D(z)} \in P[A, B].$$

In [14], Ravichandran et.al defined the class $P_n[A, B]$ as follows:

For $-1 \leq B < A \leq 1$ and

$$p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + ..., \quad n \in N,$$

we say that $p \in P_n[A, B]$ if

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \Delta.$$

The class with the property that $\frac{zf'(z)}{f(z)} \in P_n[A, B]$ is denoted by $ST_n[A, B]$. If $n = 1$, we drop the subscript. Also, Ravichandran et.al [14] obtained the following lemma:

Lemma 1.2 [14]. If $p \in P_n[A, B]$, then

$$\left| p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \leq \frac{(A - B)r^n}{1 - B^2 r^{2n}}, \quad |z| = r < 1. \tag{1.6}$$

For the special case $p \in P_n(\alpha) = P_n[1 - 2\alpha, -1]$, we get

Ref.

[1] F. M. Al-Oboudi and K. A. Al-Amoudi, Subordination results for classes of analytic functions related to conic domains defined by a fractional operator, J. Math. Anal. Appl. 354(2)(2009), 412-420..
[2] R. Aghalary, J.M. Jahangiri and S.R. Karni, Starlikeness and convexity conditions for classes of functions defined by subordination, J. Inequal. Pure Appl. Math. 5, No.2, Paper No.31, 11 p. (2004).

$$\left|p(z)-\frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}\right|\leq\frac{2(1-\alpha)r^{n}}{1-r^{2n}},\quad |z|=r<1.$$

In this paper, we define the classes:

$$\mathbf{P}=P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n;A_1,B_1,A_2,B_2,A_3,B_3,\ldots,A_N,B_N],$$

$$\mathbf{P}_n=P'^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n;A_1,B_1,A_2,B_2,A_3,B_3,\ldots,A_N,B_N];$$

Notes

of analytic functions of the single complex variable z in the unit disk $\Delta=\{z:|z|<1\}$. Moreover we study some of their basic properties. Besides we study the behavior of functions of these classes under some differential and integral operators. Concerning the class:

$$P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[A_1,B_1,A_2,B_2,A_3,B_3,\ldots,A_N,B_N],$$

which denotes the class of functions q that are analytic in Δ and are represented by

$$q(z)=\sum_{j=1}^{N}\alpha_j\left[\frac{k_j+2}{4}p_j(z)-\frac{k_j-2}{4}u_j(z)\right],$$

where $p_j,u_j\in P[A_j,B_j]$, α_j are non-negative real numbers ; $\sum_{j=1}^{\infty}\alpha_j=1$; $-1\leq B_j<A_j\leq 1,\quad k_j\geq 2$ and $j=1,2,3,\ldots,N$.

The following lemma is useful in the sequel.

Lemma 1.3 [6]. If $\psi(z)=\sum_{n=0}^{\infty}b_nz^n$ is regular in Δ, $\phi_1(z)$ and $h(z)$ are convex univalent in Δ such that $\psi(z)\prec\phi_1(z)$, then $\psi(z)*h(z)\prec\phi_1(z)*h(z),\ z\in\Delta$, where

$$\phi_1(z)=\sum_{n=0}^{\infty}a_nz^n\quad\text{and}\quad\psi(z)*\phi_1(z)=\sum_{n=0}^{\infty}b_na_nz^n.$$

II. The Class P

Suppose that

$$\mathbf{P}=P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n;A_1,B_1,A_2,B_2,A_3,B_3,\ldots,A_N,B_N]$$

denotes the class of functions q_n that are analytic in Δ and are represented by

$$q(z)=\sum_{j=1}^{N}\alpha_j\left[\frac{k_j+2}{4}p_j(z)-\frac{k_j-2}{4}u_j(z)\right],\tag{2.1}$$

where $p_j,u_j\in P_n[A_j,B_j]$, α_j are non-negative real numbers ; $\sum_{j=1}^{\infty}\alpha_j=1$; $-1\leq B_j<A_j\leq 1,\quad k_j\geq 2$ and $j=1,2,3,\ldots,N$. Since, for

$$p(z)=1+\sum_{k=n}^{\infty}a_kz^k,\quad n\in N,$$

we say that $p\in P_n[A_j,B_j]$ if $p(z)\prec\frac{1+A_jz}{1+B_jz},\ z\in\Delta$.

Lemma 2.1. The class $\mathbf{P}$ is a convex set.

Proof. We want to prove that for $\alpha,\beta\geq 0,\quad \alpha+\beta=1$ and for

$$q_1,q_2\in P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n;A_1,B_1,A_2,B_2,A_3,B_3,\ldots,A_N,B_N],$$

that $q(z)=\frac{1}{\alpha+\beta}[\alpha q_1(z)+\beta q_2(z)]$, belongs to the class

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we say that $p \in P_n[A_j, B_j]$ if $p(z) \prec \frac{1+A_j z}{1+B_j z}$, $z \in \Delta$.

Lemma 2.1. The class **P** is a convex set.

Proof. We want to prove that for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and for

$$q_1, q_2 \in P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, ..., A_N, B_N],$$

that $q(z) = \frac{1}{\alpha+\beta}[\alpha q_1(z) + \beta q_2(z)]$, belongs to the class

$$P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, ..., A_N, B_N].$$

This can simply seen by letting

$$q_1(z) = \sum_{j=1}^{N} \alpha_j \Big[\frac{k_j+2}{4} f_j(z) - \frac{k_j-2}{4} f_j^*(z)\Big],$$

where $f_j, f_j^* \in P_n[A_j, B_j]$, α_j are non-negative real numbers ; $\sum_{j=1}^{N} \alpha_j = 1$; $-1 \leq B_j < A_j \leq 1$, $k_j \geq 2$.

Also, let

$$q_2(z) = \sum_{j=1}^{N} \alpha_j \Big[\frac{k_j+2}{4} g_j(z) - \frac{k_j-2}{4} g_j^*(z)\Big],$$

where $g_j, g_j^* \in P_n[A_j, B_j]$. Then, we see that

$$\frac{1}{\alpha+\beta}[\alpha q_1(z) + \beta q_2(z)] = \sum_{j=1}^{N} \frac{\alpha_j}{\alpha+\beta} \Big[\frac{k_j+2}{4}[\alpha f_j + \beta g_j] - \frac{k_j-2}{4}[\alpha f_j^* + \beta g_j^*]\Big]$$

$$= \sum_{j=1}^{N} \alpha_j \Big[\frac{k_j+2}{4} p_j(z) - \frac{k_j-2}{4} u_j(z)\Big].$$

Then we arrive at the proof of our Lemma, since the class $P_n[A_j, B_j]$ is convex.

Lemma 2.2. Let

$$q \in P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, ..., A_N, B_N].$$

Then for

$$p(z) = 1 + \sum_{k=n}^{\infty} a_k z^k;$$

we have

(i) $|a_n| \leq \sum_{j=1}^{N} \frac{\alpha_j k_j}{2}(A_j - B_j)$ for all n.

$$(ii)\Big\{\sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})\big(1 - \frac{k_s}{2}(A_s - B_s) r^n - A_s B_s r^{2n}\big)\Big\} \Big/ \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})$$

$$\leq Rep(z)$$

$$\leq \Big\{\sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})\big(1 + \frac{k_s}{2}(A_s - B_s) r^n - A_s B_s r^{2n}\big)\Big\} \Big/ \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})$$

Notes

(iii) $q \in P_n \quad for \quad |z| < r_0$, where r_0 is the least positive root of the equation

$$\sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})\big(1 - \frac{k_s}{2}(A_s - B_s)r^n - A_s B_s r^{2n}\big) = 0, \tag{2.2}$$

and $P_n = P_n[1,-1]$ is the class of functions of positive real part. These results are sharp.

Notes

Proof. The proof of the assertion (i) is very similar to the proof of the assertion (i) of Lemma 1.1 [4]. To prove assertion (ii) of Lemma 2.2, let $p_j, u_j \in P_n[A_j, B_j]$; $-1 \leq B_j < A_j \leq 1$, $n \in N$ and $j = 1, 2, 3, ..., N$. Now, let

$$p(z) = 1 + \sum_{k=n}^{\infty} a_k z^k \prec \frac{1 + A_j z}{1 + B_j z}.$$

Then, we can write $p(z) = \frac{1+A_j\phi(z)}{1+B_j\phi(z)}$, where $\phi(z)$ is analytic in Δ, $\phi(0) = 0$ and $|\phi(z)| < 1$. Expressing $\phi(z)$ in terms of $p(z)$, we get that $\phi(z) = \frac{p(z)-1}{A_j - B_j p(z)} = \frac{a_n}{A_j - B_j} z^n + ... = z^n \Psi(z)$, where $|\Psi(z)| \leq 1$. Therefore $|\phi(z)| \leq z^n$, and hence from the subordination principle, we have that $\left|\frac{1-A_j\phi(z)}{1-B_j\phi(z)}\right| \leq Rep(z) \leq |p(z)| \leq \frac{1+A_j\phi(z)}{1+B_j\phi(z)}$, which implies that,

$$\left|\frac{1 - A_j r^n}{1 - B_j r^n}\right| \leq Rep(z) \leq |p(z)| \leq \frac{1 + A_j r^n}{1 + B_j r^n}. \tag{2.3}$$

Moreover the double inequality (2.3) will be also satisfied for the functions $u_j(z)$. Now, since

$$q \in P_{k_1,k_2,k_3,...,k_N}^{\alpha_1,\alpha_2,\alpha_3,...,\alpha_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, ..., A_N, B_N],$$

then using relation (2.1), it follows that

$$\sum_{j=1}^{N} \alpha_j \big[\frac{k_j + 2}{4} \min Rep_j(z) - \frac{k_j - 2}{4} \max u_j(z)\big] \leq Req(z) \tag{2.4}$$

$$\leq \sum_{j=1}^{N} \alpha_j \big[\frac{k_j + 2}{4} \max Rep_j(z) - \frac{k_j - 2}{4} \min u_j(z)\big].$$

Introducing the double inequality (2.3) in the double inequality (2.4), we obtain the following double inequality

$$\sum_{j=1}^{N} \alpha_j \Big\{\frac{k_j + 2}{4}\big(\frac{1 - A_j r^n}{1 - B_j r^n}\big) - \frac{k_j - 2}{4}\big[\frac{1 + A_j r^n}{1 + B_j r^n}\big]\Big\} \leq Req(z)$$

$$\leq \sum_{j=1}^{N} \alpha_j \big\{\frac{k_j + 2}{4}\big(\frac{1 + A_j r^n}{1 + B_j r^n}\big) - \frac{k_j - 2}{4}\big[\frac{1 - A_j r^n}{1 - B_j r^n}\big]\big\},$$

which yields, after simplification the required double inequality. The result of part (iii) of Lemma 2.2 follows easily from part (ii) of the same Lemma ; since

$$Req(z) \geq \left\{ \sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1; \\ j\neq s}}^{N} (1-B_j^2 r^{2n}) \left(1-\frac{k_s}{2}(A_s-B_s)r^n - A_s B_s r^{2n}\right) \right\} \Big/ \prod_{\substack{j=1; \\ j\neq s}}^{N} (1-B_j^2 r^{2n}),$$

thus Re $q(z) > 0$, for $|z| = r_0$, where r_0 is the least positive root of the equation

$$\sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1; \\ j\neq s}}^{N} (1-B_j^2 r^{2n}) \left(1-\frac{k_s}{2}(A_s-B_s)r^n - A_s B_s r^{2n}\right) = 0.$$

The function

$$q(z) = \sum_{j=1}^{N} \alpha_j \left\{ \frac{(1-\frac{k_j}{2}(A_j-B_j)z^n - A_j B_j z^{2n})}{(1-B_j^2 z^{2n})} \right\},$$

shows that the results of part (ii) and (iii) of Lemma 2.2 are sharp.

Lemma 2.3. Let

$$q \in P_{k_1,k_2,k_3,\ldots,k_N}^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_N, B_N].$$

Then

$$(i) \frac{1}{2\pi} \int_0^{2\pi} \left|q(re^{i\theta})\right|^2 d\theta \leq 1 + \frac{\left[\sum_{j=1}^{N} \frac{\alpha_j k_j}{2}(A_j - B_j)\right]^2 r^{2n}}{1-r^2},$$

$$(ii) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left|q(re^{i\theta})\right|^2 d\theta \leq \sum_{j=1}^{N} \frac{\alpha_j k_j}{2} \left[\frac{A_j - B_j}{1-B_j^2 r^{2n}}\right].$$

Proof. Let

$$q(z) = 1 + \sum_{k=n}^{\infty} a_k z^k.$$

Then by using Parseval's identity and the result of (i) given in Lemma 2.2, we get

$$\frac{1}{2\pi} \int_0^{2\pi} \left|q(re^{i\theta})\right|^2 d\theta = \sum_{k=0}^{\infty} |a_k|^2 r^{2k} \leq 1 + \sum_{k=n}^{\infty} \left[\sum_{j=1}^{N} \frac{\alpha_j k_j}{2}(A_j - B_j)\right]^2 r^{2k}$$

$$= 1 + \frac{\left[\sum_{j=1}^{N} \frac{\alpha_j k_j}{2}(A_j - B_j)\right]^2}{(1-r^2)} r^{2n}$$

Now, using relation (2.1), we get that

$$q'(z) = \sum_{j=1}^{N} \alpha_j \left[\frac{k_j+2}{4} Rep_j'(z) - \frac{k_j-2}{4} Reu_j'(z)\right]. \tag{2.5}$$

Moreover, for $p_j' \in P_n[A_j, B_j]$; we have

$$p_j'(z) = \frac{(A_j - B_j)\phi'_j(z)}{[1 + B_j \phi_j(z)]^2},$$

Notes

then

$$(2.6)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left|q(re^{i\theta})\right|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\left|(A_j-B_j)\right|\times\left|w_j'(re^{i\theta})\right|d\theta}{\left|1+B_jw_j(re^{i\theta})\right|}\le\frac{A_j-B_j}{1-B_j^2r^{2n}}.$$

Applying (2.6) in (2.5), it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|q'(re^{i\theta})\right|^2 d\theta \le \sum_{j=1}^{N}\frac{\alpha_j}{2\pi}\int_0^{2\pi}\left[\frac{k_j+2}{4}\left|p_j'(re^{i\theta})\right|+\frac{k_j-2}{4}\left|u_j'(re^{i\theta})\right|\right]d\theta$$
$$\le\sum_{j=1}^{N}\frac{\alpha_jk_j}{2}\left[\frac{A_j-B_j}{1-B_j^2r^{2n}}\right].$$

III. The Class P_N

A function f analytic in Δ is said to belong to the class

$$P'^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n;A_1,B_1,A_2,B_2,A_3,B_3,...,A_N,B_N],$$

if and only if,

$$f'\in P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n;A_1,B_1,A_2,B_2,A_3,B_3,...,A_N,B_N].$$

Lemma 3.1. The class $\mathbf{P}_n$ is a convex set.

Proof. The proof of this Lemma is very similar to the proof of Lemma 1.4 (see [6]). Now, we give the following theorem:

Theorem 3.1. Let $f\in\mathbf{P}_n$. Then f is univalent for $|z|<r_0$; where r_0 is the least positive root of the equation (2.2). This result is sharp.

Proof. Let $f\in\mathbf{P}_n$, hence it follows from Lemma 1.2.2 assertion (iii) that $Ref(z)>0$, $|z|<r_0$; where r_0 is the least positive root of the equation (2.2).

The sharpness follows from the function $f_1(z)$ defined by

$$f_1(z)=$$
$$\left\{\int_0^z\sum_{s=1}^{N}\alpha_s\prod_{j=1;j\neq s}^{N}(1-B_j^2\zeta^{2n})\big(1-\frac{k_s}{2}(A_s-B_s)\zeta^n-A_sB_s\zeta^{2n}\big)d\zeta\right\}\Big/\prod_{\substack{j=1;\\ j\neq s}}^{N}(1-B_j^2\zeta^{2n})$$

Theorem 3.2. Let $f\in\mathbf{P}_n$. Then f maps $|z|<r_1=(\sqrt{2}-1)r_0^n$ onto a convex domain, where r_0 is the least positive root of the equation (2.2). This result is sharp.

Proof. Let $f\in\mathbf{P}_n$. Hence it follows from Lemma 2.2 assertion (iii) that $Ref(z)>0$, $|z|<r_0$; where r_0 is the least positive root of the equation (2.2). Let w be any complex number such that $|w|<r_0$. Then the function

$$G(z)=P\Big(\frac{r_0^{2n}(z+w)}{r_0^{2n}+z\bar{w}}\Big)=P(w)+P'(w)\left[1-\frac{|w|^2}{r_0^{2n}}\right]z+\ldots$$

Ref.

[6] Z.M.G. Kishka, On some Classes of analytic functions, Bulletin de la Societe' Royale des Sciences de Lie'ge, 62(5-6)(1993), 313-360.

is analytic in $|z| < r_0$ and $Re\ G(z) > 0$ for $|z| < r_0$. Hence

$$\left|P'(w)(1 - \frac{|w|^2}{r_0^{2n}})\right| \leq \frac{2P(w)}{r_0^n},$$

which implies that,

$$\left|\frac{P'(w)}{P(w)}\right| \leq \frac{2r_0^n}{r_0^{2n} - |w|^2}.$$

Since w is any complex number with $|w| < r_0$, we can write the above inequality as

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{2r_0^n|z|}{r_0^{2n} - |z|^2},$$

which implies that,

$$Re(1 + \frac{zf''(z)}{f'(z)}) \geq 1 - \frac{2r_0^n|z|}{r_0^{2n} - |z|^2} = \frac{r_0^{2n} - 2r_0^n|z| - |z|^2}{r_0^{2n} - |z|^2} > 0,$$

for all $|z| < r_1 = (\sqrt{2} - 1)r_0^n$, where r_0 is the least positive root of the equation (2.2).

The function

$$f(z) = \int_0^z \frac{1 + \zeta^n}{1 - \zeta^n} d\zeta$$

shows that $(\sqrt{2} - 1)$ can not replaced by a smaller constant.

Theorem 3.3. Let $f \in \mathbf{P}_n$. Then for $z = re^{i\theta}$, we have

$$(3.1) \quad |f(z)| \geq \sum_{j=1}^{N} \alpha_j \left\{ r\left[1 - \frac{A_j k_j r^n}{2(n+1)}\right]\gamma(B_j) + \frac{A_j\Phi(B_j)}{(B_j + \gamma(B_j))} r \right.$$
$$+ \left[1 - \frac{A_j\Phi(B_j)}{(B_j + \gamma(B_j))}\right]\left[\sum_{S=0}^{\infty} \beta_j^{2s} \frac{r^{2ns+1}}{2ns+1}\right]$$
$$\left. - \frac{k_j}{2}(A_j - B_j)\Phi(B_j)\left[\sum_{S=0}^{\infty} \beta_j^{2s} \frac{r^{2ns+n+1}}{2ns+n+1}\right]\right\},$$

where,

$$\gamma(B_j) = \begin{cases} 1, & B_j = 0, \\ 0, & B_j \neq 0 \end{cases}$$

and

$$\Phi(B_j) = \begin{cases} 0, & B_j = 0, \\ 1, & B_j \neq 0. \end{cases}$$

This result is sharp for the function

$$f_0(z) = \sum_{j=1}^{N} \alpha_j \left\{ z\left[1 - \frac{A_j k_j z^n}{2(n+1)}\right]\gamma(B_j) + \frac{A_j\Phi(B_j)}{(B_j + \gamma(B_j))} z \right.$$

Notes

$$+\left[1-\frac{A_j\Phi(B_j)}{(B_j+\gamma(B_j))}\right]\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{z^{2ns+1}}{2ns+1}\right]$$
$$-\frac{k_j}{2}(A_j-B_j)\Phi(B_j)\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{z^{2ns+n+1}}{2ns+n+1}\right]\Bigg\}.$$

Proof. Since,

$$|f(z)|\geq\int_0^r Re(f'(te^{i\theta}))dt$$

Using part (ii) of Lemma 2.2, for $f'(z)=p(z)$;

$$p(z)\in P'^{\alpha_1,\alpha_2,\alpha_3,...,\alpha_N}_{k_1,k_2,k_3,...,k_N}[n;A_1,B_1,A_2,B_2,A_3,B_3,...,A_N,B_N],$$

we get that,

$$|f(z)|\geq\int_0^r\sum_{j=1}^{N}\alpha_j\left\{\frac{1-\frac{k_j}{2}(A_j-B_j)t^n-A_jB_jt^{2n}}{1-B_j^2t^{2n}}\right\}dt. \tag{3.2}$$

But

$$\frac{1-\frac{k_j}{2}(A_j-B_j)t^n-A_jB_jt^{2n}}{1-B_j^2t^{2n}}=\begin{cases}[1-\frac{k_jA_j}{2}t^n]; \quad B_j=0\\ \frac{A_j}{B_j}+\frac{[1-\frac{A_j}{B_j}]-\frac{k_j}{2}(A_j-B_j)t^n}{1-B_j^2t^{2n}};\\ \qquad B_j\neq0.\end{cases}$$

Thus

$$I=\int_0^r\frac{1-\frac{k_j}{2}(A_j-B_j)t^n-A_jB_jt^{2n}}{1-B_j^2t^{2n}}dt$$

$$=\begin{cases}[1-\frac{k_jA_j}{2(n+1)}r^n]r; \qquad B_j=0\\ \frac{A_j}{B_j}r+\left\{(1-\frac{A_j}{B_j})\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{r^{2ns+1}}{2ns+1}\right]-\frac{k_j}{2}(A_j-B_j)\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{r^{2ns+n+1}}{2ns+n+1}\right]\right\},\\ s=1,2,...,N;\ B_j\neq0,\end{cases}$$

which implies that,

$$I=\left[1-\frac{k_jA_j}{2(n+1)}r^n\right]r\gamma(B_j)+\frac{A_j\Phi(B_j)}{(B_j+\gamma(B_j))}r \tag{3.3}$$
$$+\left[1-\frac{A_j\Phi(B_j)}{(B_j+\gamma(B_j))}\right]\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{r^{2ns+1}}{2ns+1}\right]-\frac{k_j}{2}(A_j-B_j)\Phi(B_j)\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{r^{2ns+n+1}}{2ns+n+1}\right].$$

Introducing (3.3) in the right hand side of inequality (3.2), we obtain inequality (3.1).

Remark 3.1. If we put $n=1$ in Theorems 3.1, 3.2 and 3.3, we obtain the corresponding results in [6].

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